Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

Finite element method (FEM1)

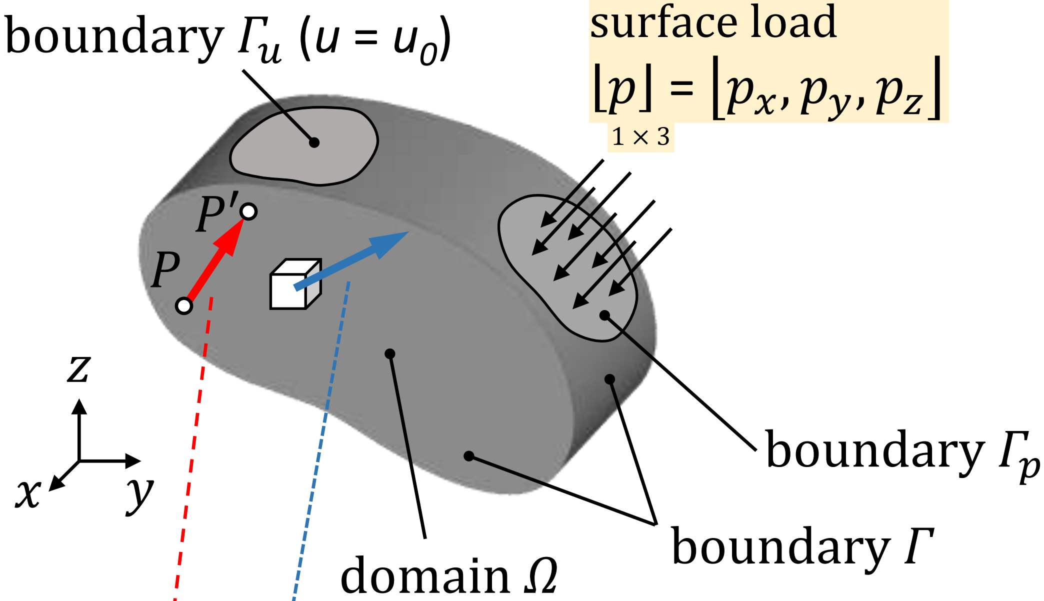
Lecture 2A. The boundary value problem of solid mechanics in the FEM approach

03.2025

Boundary value problem of solid body mechanics

surface load

$\lfloor p \rfloor_{1\times 3} = \lfloor p_x, p_y, p_z \rfloor$



mass forces

$\lfloor X \rfloor_{1\times 3} = \lfloor X, Y, Z \rfloor = \rho \lfloor a_x, a_y, a_z \rfloor$

accelerations

UNKNOWN FUNCTION

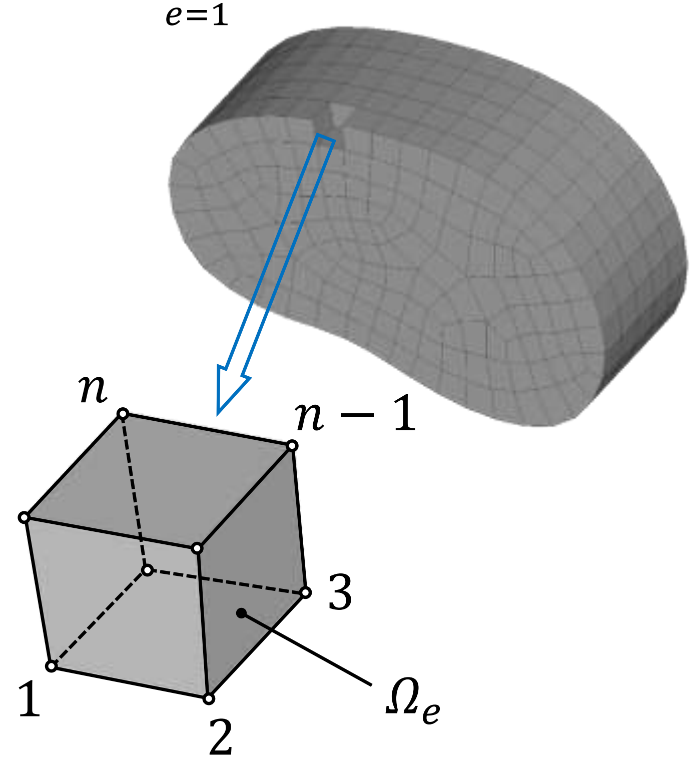
Displacement vector $\underset{3\times 1}{\{u\}} = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix}$

FE model

NOE – no. of FEs

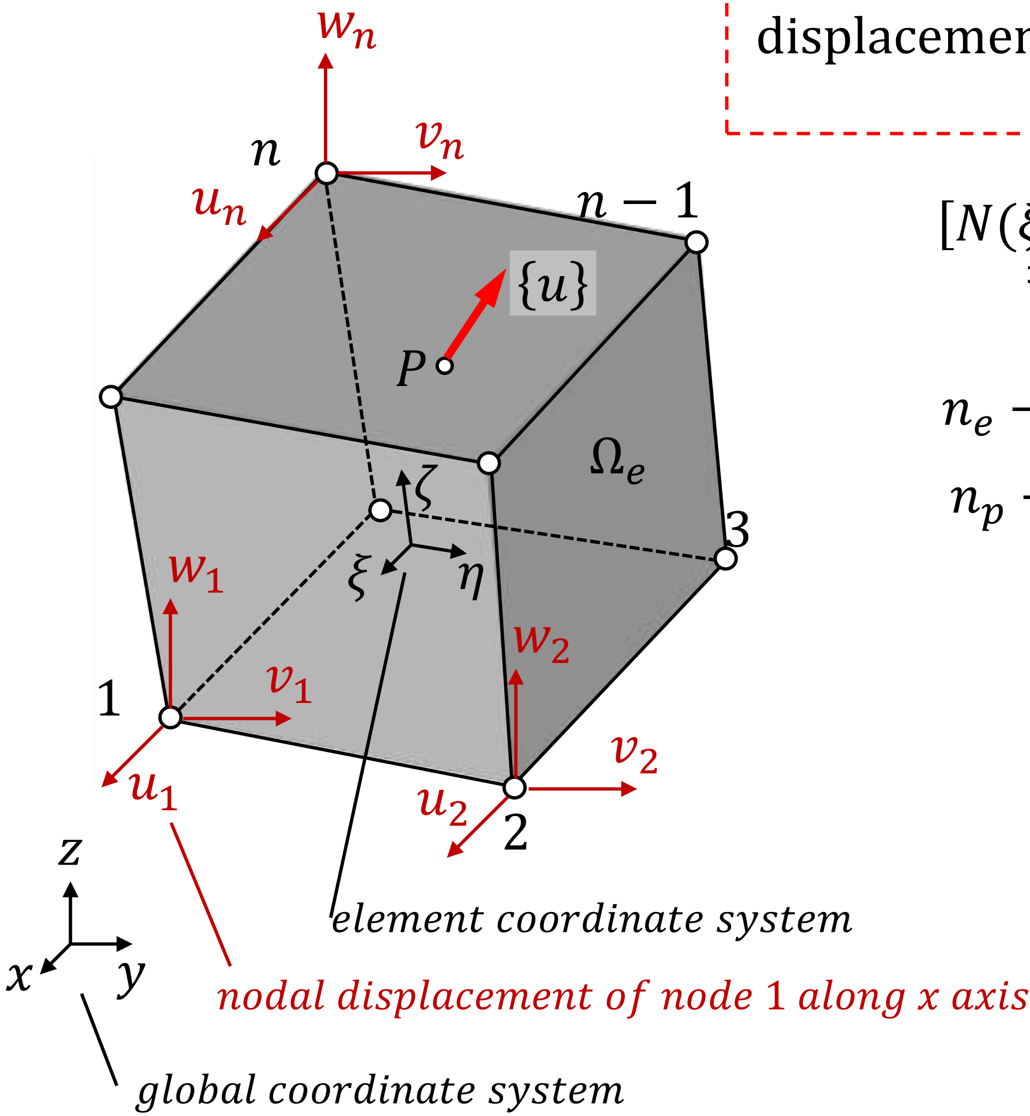
NON – no. of nodes

$$\Omega = \sum_{e=1}^{NOE} \Omega_e \ \text{ and } \ \Omega_i \underset{i\neq j}{\cap} \Omega_j = 0$$



Finite element with n - nodes

Nodal approximation inside the finite element with n - nodes



displacement vector $\underset{3\times 1}{\{u\}} = \underset{3\times n_e}{[N(\xi,\eta,\zeta)]}\underset{n_e\times 1}{\{q\}_e}$

$\underset{3\times n_e}{[N(\xi,\eta,\zeta)]}$ – matrix of shape funtions

$$n_e = n \cdot n_p$$

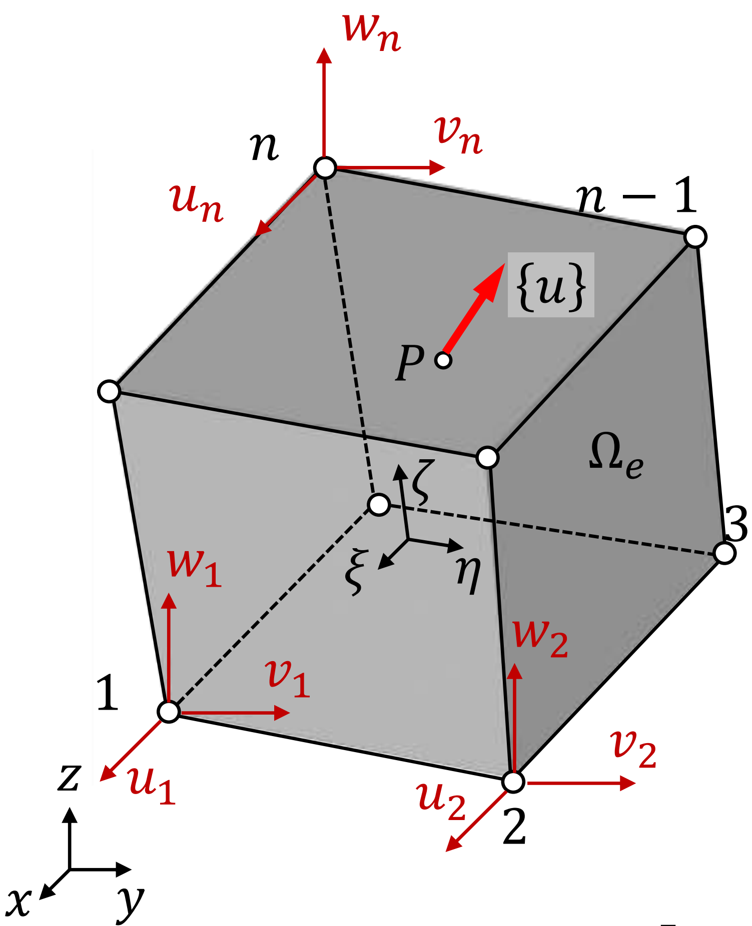
n_e – no. of degrees of freedom in FE

n_p – no. of degrees of freedom per node

$$\underset{n_e\times 1}{\{q\}_e} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix}_e$$

– local vector of nodal parameters

Matrix of shape functions



Nodal approximation:

$$\underset{3\times 1}{\{u\}} = \underset{3\times n_e}{[N(\xi,\eta,\zeta)]}\underset{n_e\times 1}{\{q\}_e}$$

$$u = N_1 \cdot u_1 + N_2 \cdot u_2 + \ldots + N_n \cdot u_n$$
$$v = N_1 \cdot v_1 + N_2 \cdot v_2 + \ldots + N_n \cdot v_n$$
$$w = N_1 \cdot w_1 + N_2 \cdot w_2 + \ldots + N_n \cdot w_n$$

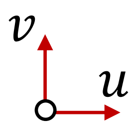
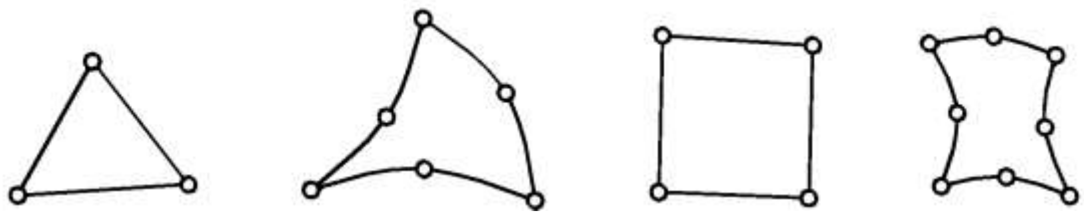
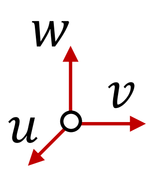
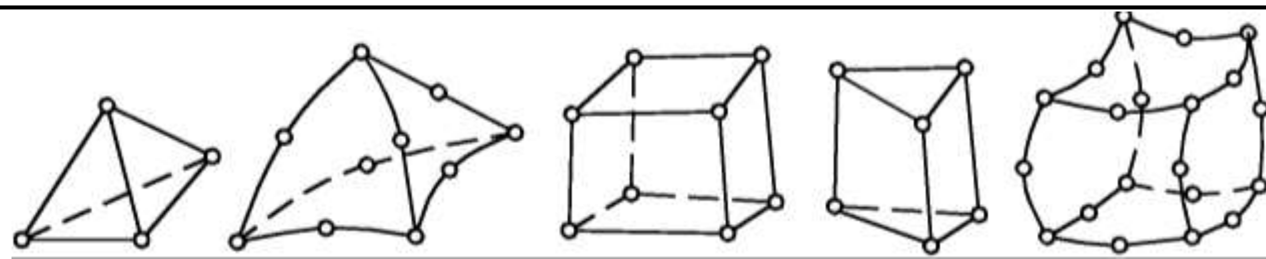
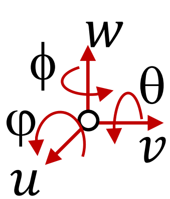
Vector of element nodal parameters:

$$\underset{n_e\times 1}{\{q\}_e} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix}_e$$

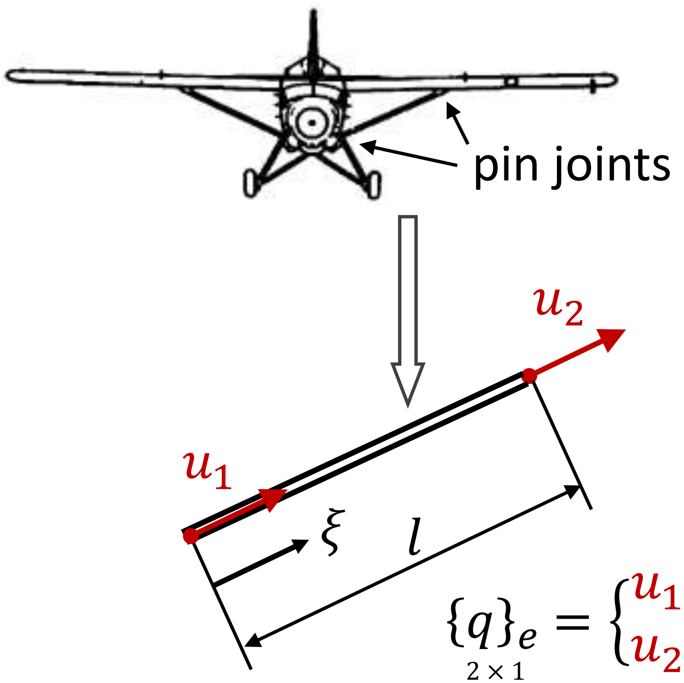
Shape function matrix:

$$\underset{3\times n_e}{[N(\xi,\eta,\zeta)]} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \ldots & 0 & N_n & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & & 0 & 0 & N_n \end{bmatrix}$$

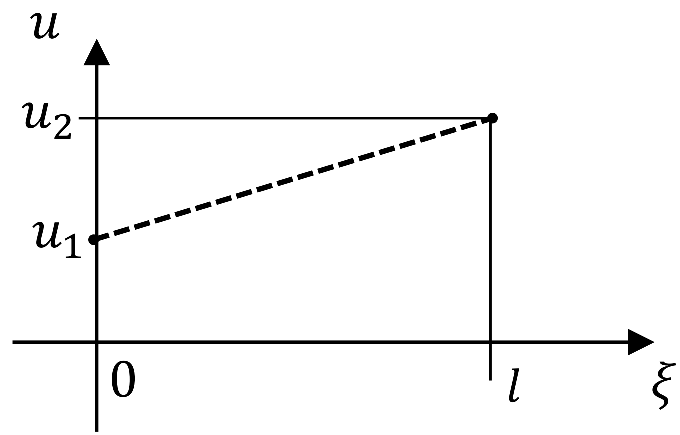
Examples of finite elements

Type	n_e − number of degrees of freedom in FE
rods	 2, 4, 6 6
2D 	 6 12 8 16
3D 	 12 30 24 18 60
shell 	 24 48

Example 1: shape functions for a finite element representing a strut



$$\underset{2\times 1}{\{q\}_e} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e$$



linear function:

$$u(\xi) = \frac{u_2 - u_1}{l}\xi + u_1$$

$$u(\xi) = \frac{u_2 - u_1}{l}\xi + u_1 = \frac{u_2}{l}\xi - \frac{u_1}{l}\xi + u_1 = \left(1 - \frac{\xi}{l}\right)u_1 + \frac{\xi}{l}u_2 =$$

$$= N_1(\xi)\cdot u_1 + N_2(\xi)\cdot u_2 = \lfloor N_1, N_2 \rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e = \underset{1\times 2}{\lfloor N(\xi) \rfloor}\underset{2\times 1}{\{q\}_e}$$

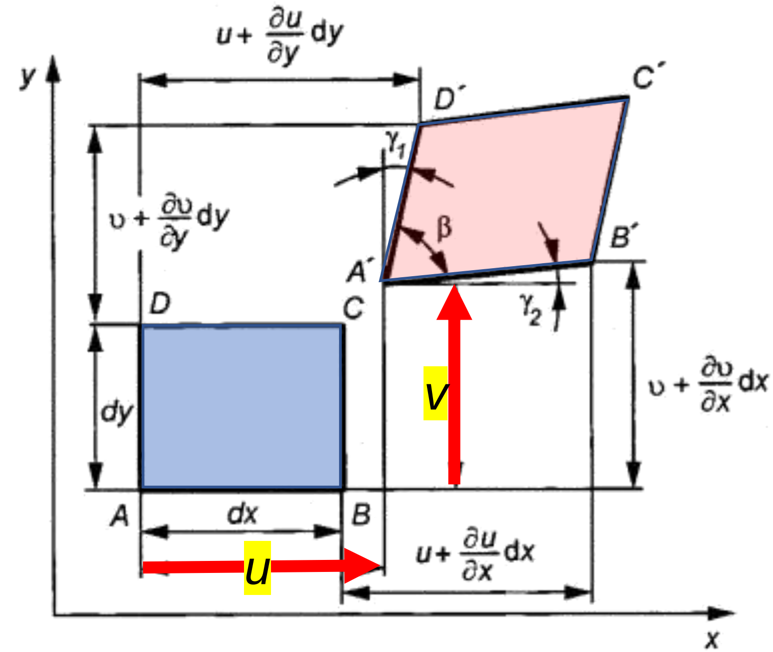
shape functions: $\boxed{N_1(\xi) = 1 - \frac{\xi}{l}}$; $\boxed{N_2(\xi) = \frac{\xi}{l}}$

Strain components

normal strains:

$$\varepsilon_x = \frac{(A'B')_x - AB}{AB} = \frac{(\mathrm{dx}+u+\frac{\partial u}{\partial x}\,\mathrm{dx}-u)-\mathrm{dx}}{\mathrm{dx}} = \boxed{\frac{\partial u}{\partial x}}$$

$$\boxed{\varepsilon_y = \frac{\partial v}{\partial y}} \quad ; \quad \boxed{\varepsilon_z = \frac{\partial w}{\partial z}}$$



shear strains:

$$\gamma_{xy} = \frac{\pi}{2} - \beta = \gamma_1 + \gamma_2$$

$$\gamma_1 \cong \tan\gamma_1 = \frac{(A'D')_x}{(A'D')_y} = \frac{u+\frac{\partial u}{\partial y}\,\mathrm{dy}-u}{\mathrm{dy}+v+\frac{\partial v}{\partial y}\,\mathrm{dy}-v} = \frac{\frac{\partial u}{\partial y}}{1+\frac{\partial v}{\partial y}} = \frac{\frac{\partial u}{\partial y}}{1+\varepsilon_y} = \frac{\partial u}{\partial y}$$

$$\gamma_2 \cong \frac{\partial v}{\partial x} \quad \rightarrow \quad \boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}}$$

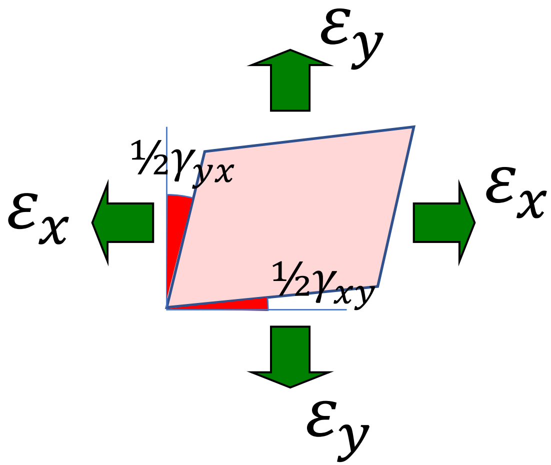
small deformations : $\varepsilon_y \ll 1$

$$\boxed{\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}} \quad ; \quad \boxed{\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}} \quad ; \quad \gamma_{ij} = \gamma_{ji}$$

Strain tensor. Vector of strain components

strain tensor:

$$\underset{3\times 3}{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \varepsilon_y & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \varepsilon_z \end{bmatrix}$$



vector of strain components:

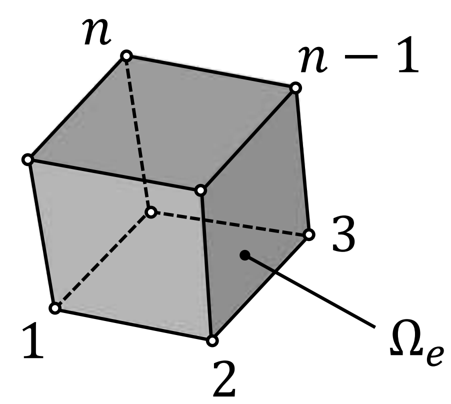
$$\underset{6\times 1}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \underset{6\times 3}{[R]}\underset{3\times 1}{\{u\}} \quad ; \quad \underset{1\times 6}{\lfloor \varepsilon \rfloor} = \underset{1\times 3}{\lfloor u \rfloor}\underset{3\times 6}{[R]^T}$$

gradient matrix

Strain – displacement matrix of a finite element

nodal approximation in a finite element:

$$\underset{3\times 1}{\{u\}} = \underset{3\times n_e}{[N(\xi,\eta,\zeta)]}\underset{n_e\times 1}{\{q\}_e}$$



vector of strain components in a finite element:

$$\underset{6\times 1}{\{\varepsilon\}} = \underset{6\times 3}{[R]}\underset{3\times 1}{\{u\}} = \underset{6\times 3}{[R]}\underset{3\times n_e}{[N]}\underset{n_e\times 1}{\{q\}_e} = \underset{6\times n_e}{[B]}\underset{n_e\times 1}{\{q\}_e}$$

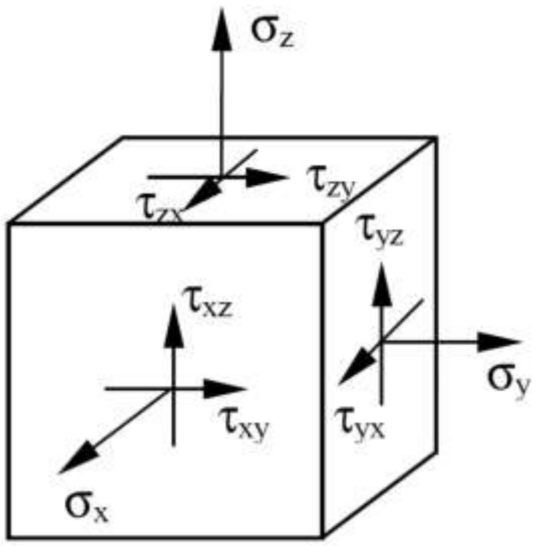
$$\underset{1\times 6}{\lfloor\varepsilon\rfloor} = \underset{1\times n_e}{\lfloor q\rfloor_e}\underset{n_e\times 6}{[B]^T}$$

$$\underset{6\times n_e}{[B]} = \underset{6\times 3}{[R]}\underset{3\times n_e}{[N]} \quad - \textit{strain–displacement matrix}$$

Stress components

normal stresses: $\sigma_x \;\; ; \;\; \sigma_y \;\; ; \;\; \sigma_z$

positive value - tension, negative value - compression



shear stress components:

$\tau_{xy} \;\; ; \;\; \tau_{yz} \;\; ; \;\; \tau_{zx} \;\; ; \;\; \tau_{ij} = \tau_{ji}$

equivalent stresses:

Von Mises stress:

$$\sigma_{EQV} = \sqrt{\frac{1}{2}\left(\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2\right) + 3({\tau_{xy}}^2 + {\tau_{yz}}^2 + {\tau_{zx}}^2)}$$

Tresca stress: $\sigma_{INT} = \sigma_1 - \sigma_3 = 2\tau_{max}$

the first principal stress *the third principal stress* *maximum shear stress*

Stress tensor. Vector of stress components

stress tensor:

$$\underset{3\times 3}{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \equiv \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

in coordinate system x, y, z

in the principal coordinate system

vector of stress components:

$$\underset{6\times 1}{\{\sigma\}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

Constitutive matrix

linear isotropic material (Hooke's law):

$$\underset{6\times 1}{\{\sigma\}} = \underset{6\times 6}{[D]}\ \underset{6\times 1}{\{\varepsilon\}}$$

constitutive matrix:

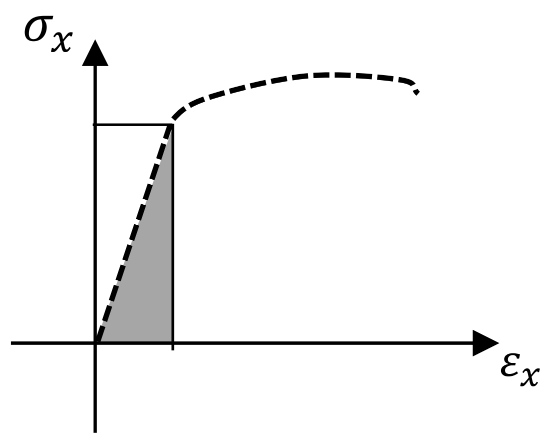
$$\underset{6\times 6}{[D]} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix}$$

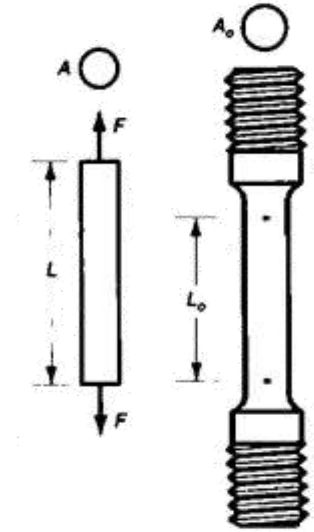
E – Young's modulus,
ν – Poisson's ratio

Example 2: uniaxial tensile test

$$\sigma_x = \frac{F}{A_0} \quad ; \quad \varepsilon_x = \frac{L-L_0}{L_0} \quad ; \quad \varepsilon_y = \varepsilon_z = \varepsilon_T$$

elastic strain Energy: $U = \frac{1}{2}\sigma_x\, \varepsilon_x\, A_0\, L_0$





$$\underset{6\times 1}{\{\sigma\}} = \underset{6\times 6}{[D]}\ \underset{6\times 1}{\{\varepsilon\}}$$

$$\begin{Bmatrix} \sigma_x \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-v \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_T \\ \varepsilon_T \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

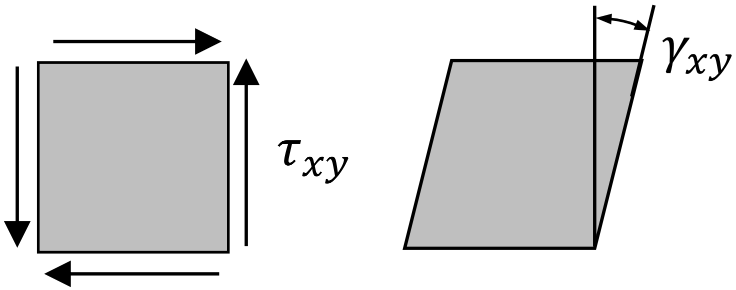
<u>2nd equation :</u> $0 = \frac{E}{(1+\nu)(1-2\nu)}(v\varepsilon_x + (1-v)\varepsilon_T + v\varepsilon_T) \rightarrow \boxed{\varepsilon_T = -v\varepsilon_x}$

<u>1st equation:</u>

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}((1-v)\varepsilon_x + v\varepsilon_T + v\varepsilon_T) = \frac{E}{(1-\nu-2v^2)}((1-v)\varepsilon_x - v^2\varepsilon_x - v^2\varepsilon_x) \rightarrow \boxed{\sigma_x = E\,\varepsilon_x}$$

Example 3: pure shear

τ_{xy} ; γ_{xy}



$$\underset{6\times1}{\{\sigma\}} = \underset{6\times6}{[D]}\ \underset{6\times1}{\{\varepsilon\}}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \tau_{xy} \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \gamma_{xy} \\ 0 \\ 0 \end{Bmatrix}$$

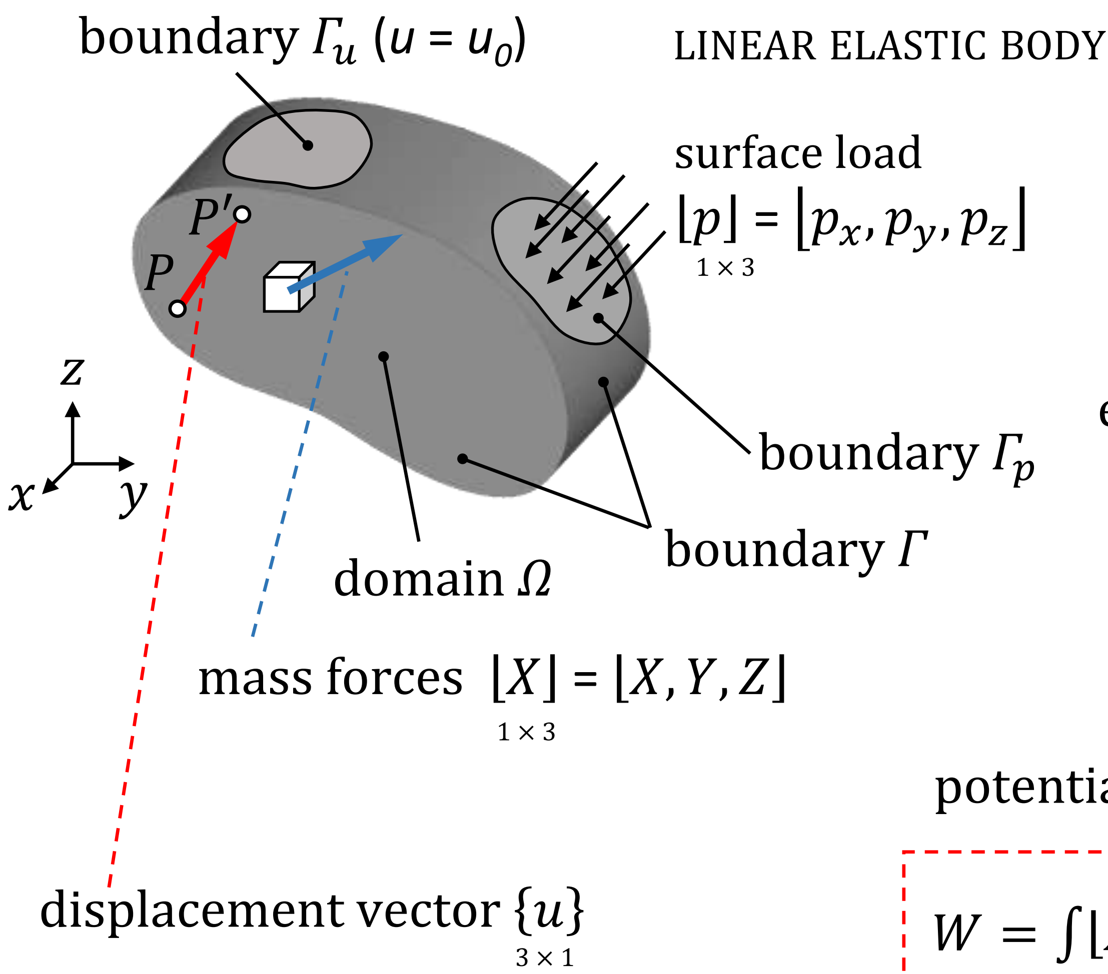
4th equation:

$$\tau_{xy} = \frac{E}{(1+\nu)(1-2\nu)}(0.5-\nu)\gamma_{xy} = \frac{E}{2(1+\nu)(0.5-\nu)}(0.5-\nu)\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \rightarrow$$

$$\boxed{\tau_{xy} = G\,\gamma_{xy}}$$

$G = \frac{E}{2(1+\nu)}$ – *Kirchof's modulus (shear modulus)*

Elastic strain energy. Potential energy of loading



elastic strain energy:

$$U = \frac{1}{2}\int_{\Omega} \underset{1\times 6}{\lfloor \varepsilon \rfloor}\underset{6\times 1}{\{\sigma\}}\, d\Omega$$

potential energy of loading:

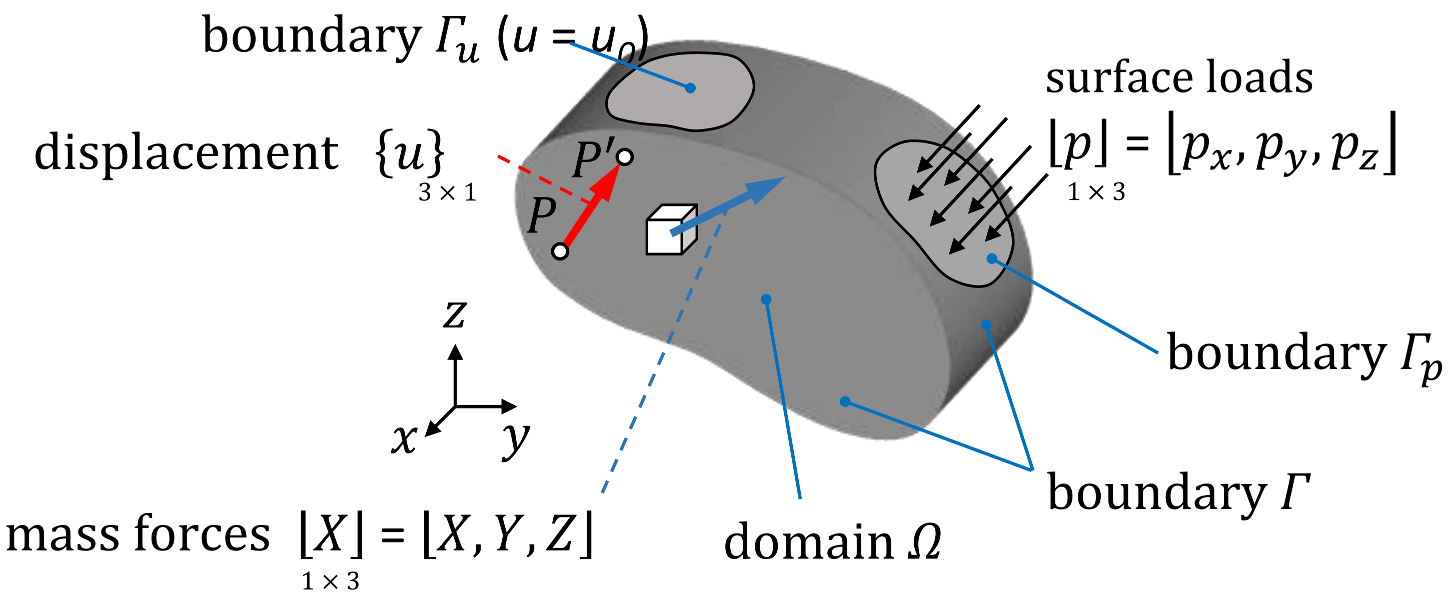
$$W = \int_{\Omega} \underset{1\times 3}{\lfloor X \rfloor}\underset{3\times 1}{\{u\}}\, d\Omega + \int_{\Gamma_p} \underset{1\times 3}{\lfloor p \rfloor}\underset{3\times 1}{\{u\}}\, d\Gamma_p$$

Minimum total potential energy principle

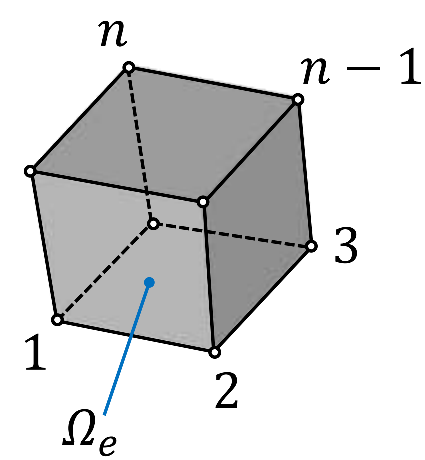
total potential energy: $V = U - W$

The displacement field $\{u\}$ that represents solution of the problem fulfils displacement boundary conditions on Γ_u and minimizes the total potential energy V.

$$\boxed{V \rightarrow min}$$



Elastic strain energy in a finite element. Local stiffness matrix



$\underset{n_e\times 1}{\{q\}_e}$ - local vector of nodal parameters

elastic strain energy in a finite element:

$$U_e = \frac{1}{2}\int\limits_{\Omega_e} \underset{1\times 6}{\lfloor\varepsilon\rfloor}\underset{6\times 1}{\{\sigma\}}\, d\Omega_e = \frac{1}{2}\underset{1\times n_e}{\lfloor q\rfloor_e}\int\limits_{\Omega_e}\underset{n_e\times 6}{[B]^T}\underset{6\times 6}{[D]}\underset{6\times n_e}{[B]}\, d\Omega_e\,\underset{n_e\times 1}{\{q\}_e} = \frac{1}{2}\underset{1\times n_e}{\lfloor q\rfloor_e}\underset{n_e\times n_e}{[k]_e}\underset{n_e\times 1}{\{q\}_e}$$

$$\underset{6\times 1}{\{\sigma\}} = \underset{6\times 6}{[D]}\ \underset{6\times 1}{\{\varepsilon\}}$$

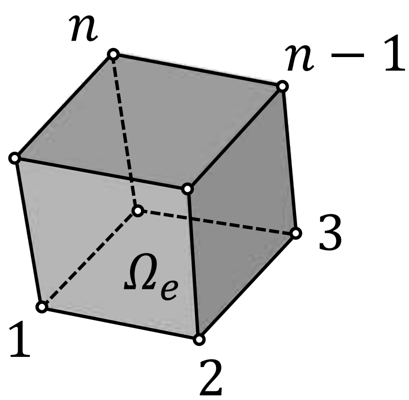
$$\underset{1\times 6}{\lfloor\varepsilon\rfloor} = \underset{1\times n_e}{\lfloor q\rfloor_e}\underset{n_e\times 6}{[B]^T} \qquad \underset{6\times 1}{\{\varepsilon\}} = \underset{6\times n_e}{[B]}\underset{n_e\times 1}{\{q\}_e}$$

local stiffness matrix:

$$\boxed{\underset{n_e\times n_e}{[k]_e} = \int\limits_{\Omega_e}\underset{n_e\times 6}{[B]^T}\underset{6\times 6}{[D]}\underset{6\times n_e}{[B]}d\Omega_e}$$

Elastic strain energy in a finite element

local notation:



n – no. of nodes per FE

n_p – no. of nodal parameters per node

no. of degrees of freedom in FE:

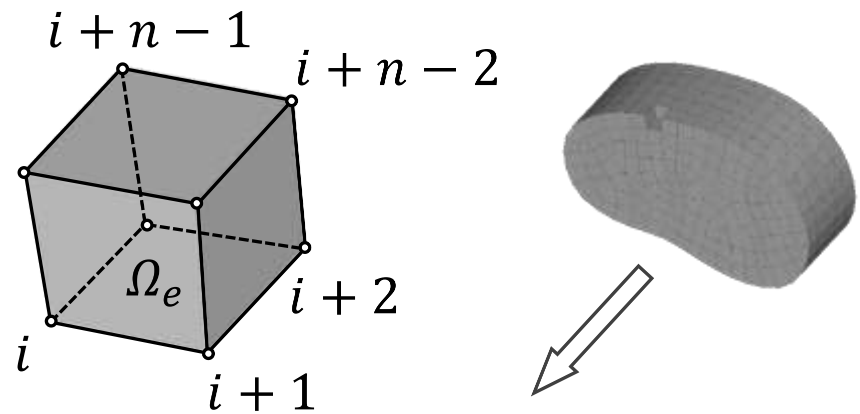
$$n_e = n \cdot n_p$$

$\underset{n_e \times 1}{\{q\}_e}$ - local vector of nodal parameters

$$U_e = \frac{1}{2} \underset{1 \times n_e}{\lfloor q \rfloor_e} \underset{n_e \times n_e}{[k]_e} \underset{n_e \times 1}{\{q\}_e}$$

local stiffness matrix

global notation:



NON – no. of nodes

n_p – no. of nodal parameters per node

no. of degrees of freedom :

$$NDOF = NON \cdot n_p$$

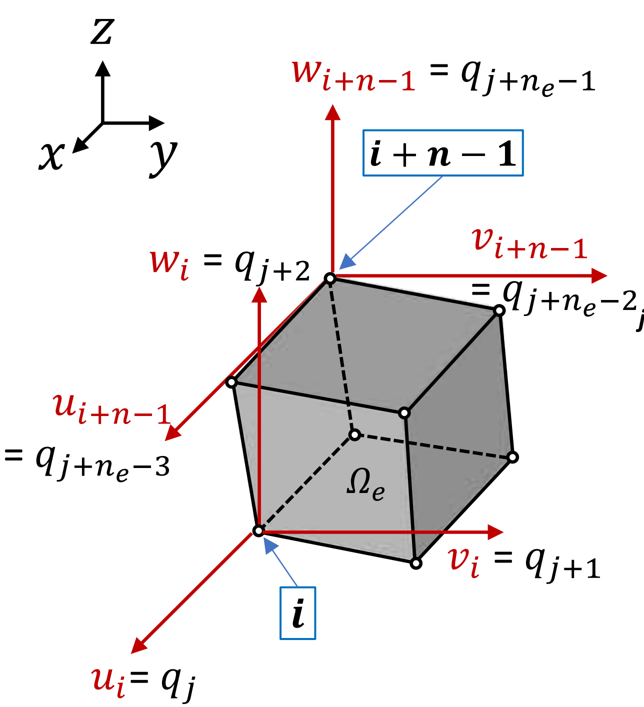
$\underset{NDOF \times 1}{\{q\}}$ - global vector of nodal parameters

$$U_e = \frac{1}{2} \cdot \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times NDOF}{[k]_e^*} \cdot \underset{NDOF \times 1}{\{q\}}$$

extended local stiffness matrix.

Extended local stiffness matrix of a finite element

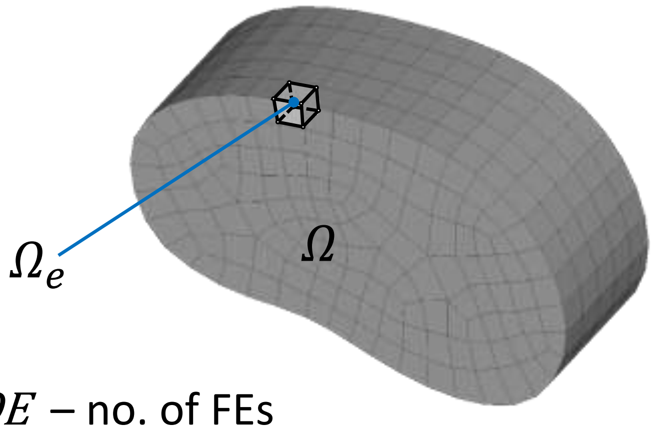
$$\underset{NDOF\times 1}{\{q\}} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_j \\ \vdots \\ q_{NDOF} \end{Bmatrix}$$



$$[k]_e^* = \begin{array}{c} \\ 1 \\ 2 \\ \dots \\ j-1 \\ j \\ j+1 \\ \dots \\ j+n_e-1 \\ j+n_e \\ \dots \\ NDOF \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & \dots & j-1 & j & j+1 & \dots & j+n_e-1 & j+n_e & \dots & NDOF \end{array} \\ \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & k_{11} & k_{12} & \dots & k_{1n_e} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & k_{21} & k_{22} & \dots & k_{2n_e} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & k_{n_e1} & k_{n_e2} & \dots & k_{n_en_e} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(assumed ascending order of components)

Elastic strain energy in a FE model. Global stiffness matrix



NOE – no. of FEs

$NDOF$ – no. of degrees of freedom

$$\Omega = \sum_{e=1}^{NOE} \Omega_e \quad \rightarrow \quad \boxed{U = \sum_{e=1}^{NOE} U_e}$$

$\underset{NDOF \times 1}{\{q\}}$ - global vector of nodal parameters

elastic strain energy in a finite element model:

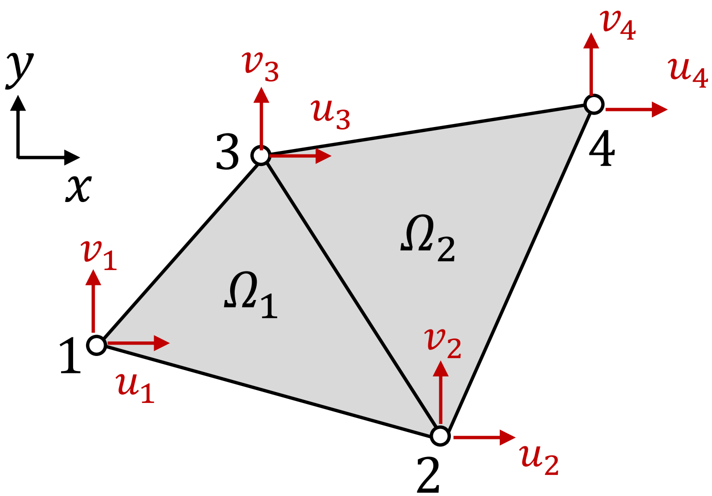
$$U = \sum_{e=1}^{NOE} U_e = \sum_{e=1}^{NOE} \frac{1}{2} \cdot \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times NDOF}{[k]_e^*} \cdot \underset{NDOF \times 1}{\{q\}} = \frac{1}{2} \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \sum_{e=1}^{NOE} \underset{NDOF \times NDOF}{[k]_e^*} \cdot \underset{NDOF \times 1}{\{q\}} =$$

$$= \frac{1}{2} \cdot \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times NDOF}{[K]} \cdot \underset{NDOF \times 1}{\{q\}}$$

global stiffness matrix:

$$\boxed{\underset{NDOF \times NDOF}{[K]} = \sum_{e=1}^{NOE} [k]_e^*}$$

Example 4: global stiffness matrix of a 2D model with two 3-node triangles



global notation:

$NOE = 2$

$NON = 4$

$n = 3$

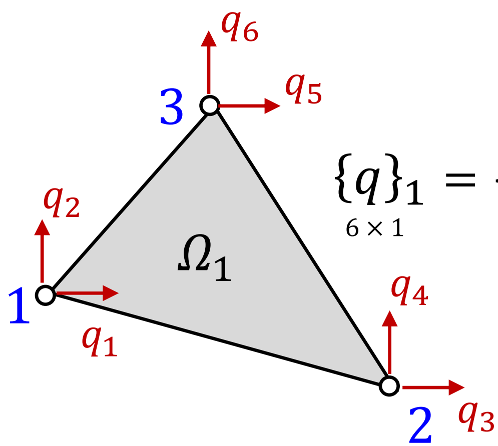
$n_p = 2 \quad ; \quad (u, v)$

$n_e = n \cdot n_p = 6$

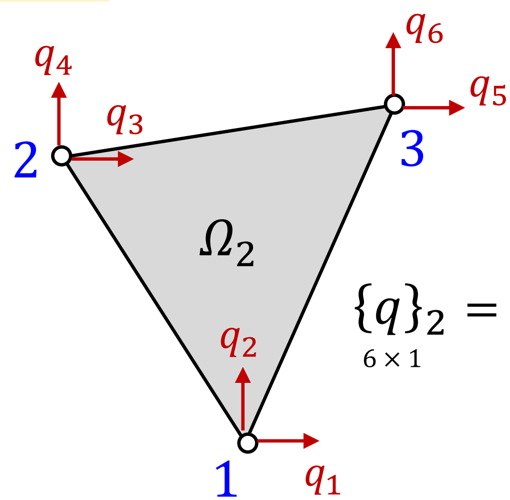
$NDOF = NON \cdot n_p = 8$

$$\underset{8\times 1}{\{q\}} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

local notation:

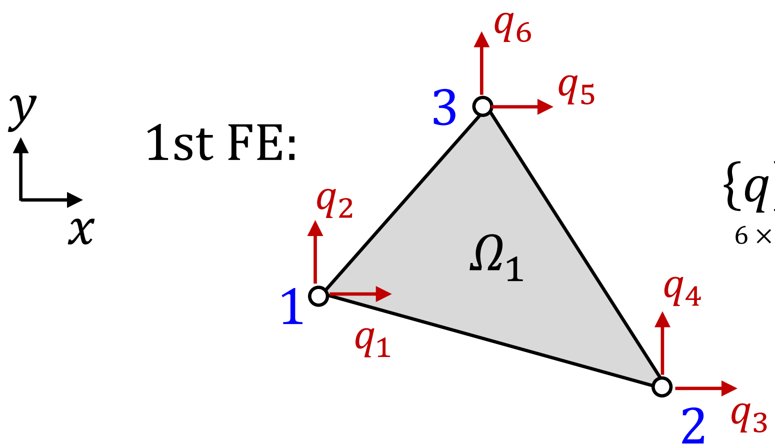


$$\underset{6\times 1}{\{q\}_1} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_1 = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}_1$$



$$\underset{6\times 1}{\{q\}_2} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_2 = \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}_2$$

Example 4: global stiffness matrix of a 2D model with two 3-node triangles

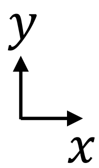


$$\underset{6\times 1}{\{q\}_1} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_1 = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}_1 \Rightarrow \underset{8\times 1}{\{q\}} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

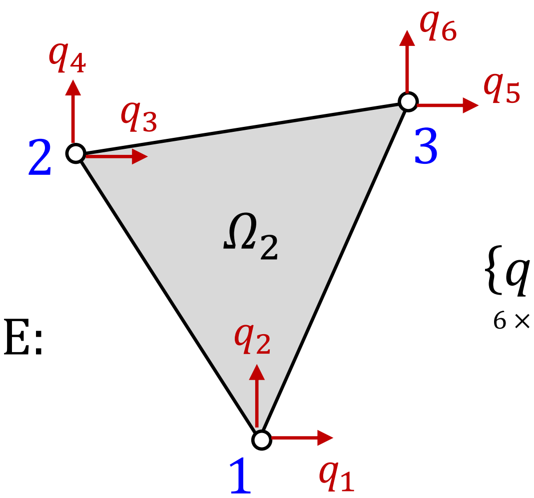
$$\underset{6\times 6}{[k]_1} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ b_1 & g_1 & h_1 & i_1 & j_1 & k_1 \\ c_1 & h_1 & l_1 & m_1 & n_1 & o_1 \\ d_1 & i_1 & m_1 & p_1 & r_1 & s_1 \\ e_1 & j_1 & n_1 & r_1 & t_1 & \bar{u}_1 \\ f_1 & k_1 & o_1 & s_1 & \bar{u}_1 & \bar{w}_1 \end{bmatrix} \end{array}$$

$$\underset{8\times 8}{[k]_1^*} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & 0 & 0 \\ b_1 & g_1 & h_1 & i_1 & j_1 & k_1 & 0 & 0 \\ c_1 & h_1 & l_1 & m_1 & n_1 & o_1 & 0 & 0 \\ d_1 & i_1 & m_1 & p_1 & r_1 & s_1 & 0 & 0 \\ e_1 & j_1 & n_1 & r_1 & t_1 & \bar{u}_1 & 0 & 0 \\ f_1 & k_1 & o_1 & s_1 & \bar{u}_1 & \bar{w}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 4: global stiffness matrix of a 2D model with two 3-node triangles



2nd FE:

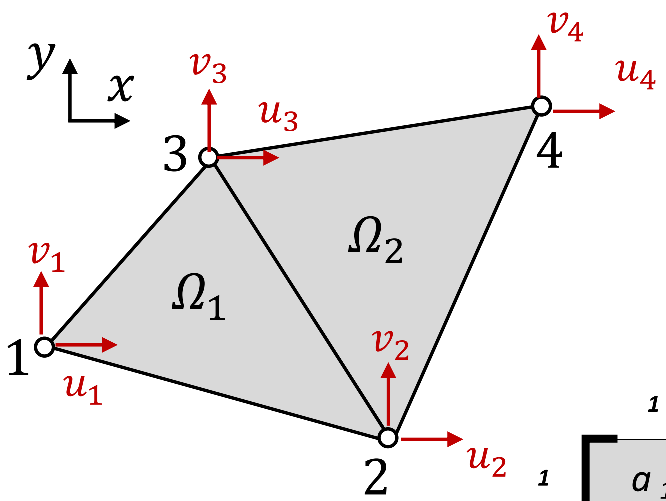


$$\underset{6\times 1}{\{q\}_2} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_2 = \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}_2 \qquad \underset{8\times 1}{\{q\}} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\underset{6\times 6}{[k]_2} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ b_2 & g_2 & h_2 & i_2 & j_2 & k_2 \\ c_2 & h_2 & l_2 & m_2 & n_2 & o_2 \\ d_2 & i_2 & m_2 & p_2 & r_2 & s_2 \\ e_2 & j_2 & n_2 & r_2 & t_2 & \bar{u}_2 \\ f_2 & k_2 & o_2 & s_2 & \bar{u}_2 & \bar{w}_2 \end{bmatrix} \end{array}$$

$$\underset{8\times 8}{[k]_2^*} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ 0 & 0 & b_2 & g_2 & h_2 & i_2 & j_2 & k_2 \\ 0 & 0 & c_2 & h_2 & l_2 & m_2 & n_2 & o_2 \\ 0 & 0 & d_2 & i_2 & m_2 & p_2 & r_2 & s_2 \\ 0 & 0 & e_2 & j_2 & n_2 & r_2 & t_2 & \bar{u}_2 \\ 0 & 0 & f_2 & k_2 & o_2 & s_2 & \bar{u}_2 & \bar{w}_2 \end{bmatrix} \end{array}$$

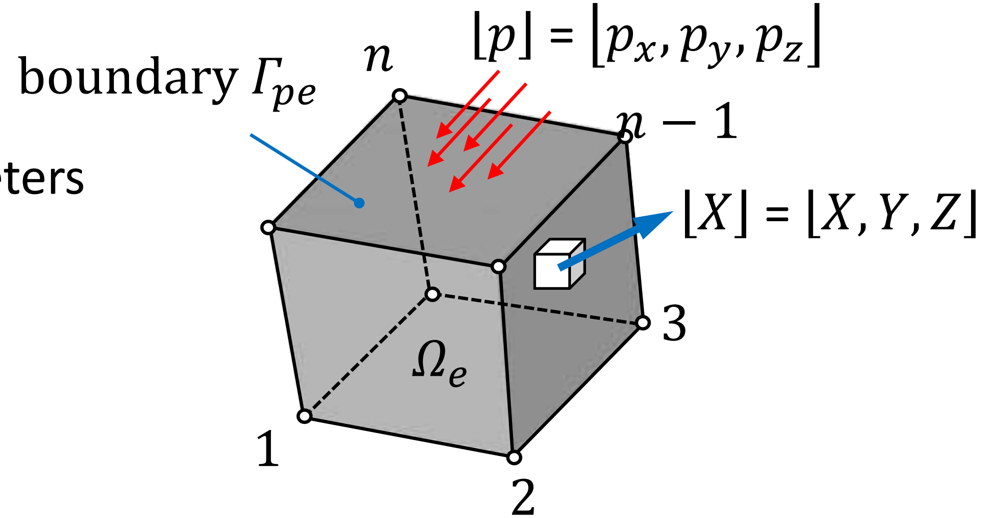
Example 4: global stiffness matrix of a 2D model with two 3-node triangles



$$\underset{8\times 1}{\{q\}} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\underset{8\times 8}{[K]} = \underset{8\times 8}{[k]_1^*} + \underset{8\times 8}{[k]_2^*} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & 0 & 0 \\ b_1 & g_1 & h_1 & i_1 & j_1 & k_1 & 0 & 0 \\ c_1 & h_1 & l_1+a_2 & m_1+b_2 & n_1+c_2 & o_1+d_2 & e_2 & f_2 \\ d_1 & i_1 & m_1+b_2 & p_1+g_2 & r_1+h_2 & s_1+i_2 & j_2 & k_2 \\ e_1 & j_1 & n_1+c_2 & r_1+h_2 & t_1+l_2 & \bar{u}_1+m_2 & n_2 & o_2 \\ f_1 & k_1 & o_1+d_2 & t_1+l_2 & \bar{u}_1+m_2 & \bar{w}_1+p_2 & r_2 & s_2 \\ 0 & 0 & e_2 & j_2 & n_2 & r_2 & t_2 & \bar{u}_2 \\ 0 & 0 & f_2 & k_2 & o_2 & s_2 & \bar{u}_2 & \bar{w}_2 \end{bmatrix} \end{array}$$

Potential energy of loading in a finite element



$\underset{n_e \times 1}{\{q\}_e}$ - local vector of nodal parameters

potential energy of loading
in a finite element :

$$W_e = \int_{\Omega_e} \underset{1\times 3}{\lfloor X \rfloor} \underset{3\times 1}{\{u\}}\, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 3}{\lfloor p \rfloor} \underset{3\times 1}{\{u\}}\, d\Gamma_{pe} = \int_{\Omega_e} \underset{1\times 3}{\lfloor X \rfloor} \underset{3\times n_e}{[N]} \underset{n_e\times 1}{\{q\}_e}\, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 3}{\lfloor p \rfloor} \underset{3\times n_e}{[N]} \underset{n_e\times 1}{\{q\}_e}\, d\Gamma_{pe} =$$

$$\underset{3\times 1}{\{u\}} = \underset{3\times n_e}{[N]} \underset{n_e\times 1}{\{q\}_e}$$

$$= \left(\int_{\Omega_e} \underset{1\times 3}{\lfloor X \rfloor} \underset{3\times n_e}{[N]}\, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 3}{\lfloor p \rfloor} \underset{3\times n_e}{[N]}\, d\Gamma_{pe} \right) \underset{n_e\times 1}{\{q\}_e} = \left(\underset{1\times n_e}{\lfloor F^X \rfloor_e} + \underset{1\times n_e}{\lfloor F^p \rfloor_e} \right) \underset{n_e\times 1}{\{q\}_e} = \underset{1\times n_e}{\lfloor F \rfloor_e} \underset{n_e\times 1}{\{q\}_e}$$

equivalent load vector:

$$\boxed{\underset{1\times n_e}{\lfloor F \rfloor_e} = \underset{1\times n_e}{\lfloor F^X \rfloor_e} + \underset{1\times n_e}{\lfloor F^p \rfloor_e}}$$

Equivalent load vector

$$\underset{1\times n_e}{\lfloor F \rfloor_e} = \underset{1\times n_e}{\lfloor F^X \rfloor_e} + \underset{1\times n_e}{\lfloor F^p \rfloor_e}$$

equivalent load vector due to mass forces:

$$\boxed{\underset{1\times n_e}{\lfloor F^X \rfloor_e} = \int_{\Omega_e} \underset{1\times 3}{\lfloor X \rfloor} \underset{3\times n_e}{[N]} d\Omega_e} =$$

$$= \int_{\Omega_e} \lfloor X, Y, Z \rfloor \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots & 0 & N_n & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & & 0 & 0 & N_n \end{bmatrix} d\Omega_e$$

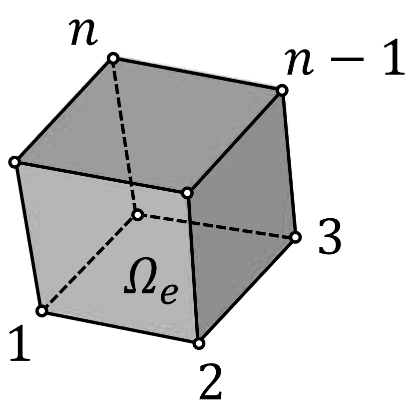
equivalent load vector due to surface load:

$$\boxed{\underset{1\times n_e}{\lfloor F^p \rfloor_e} = \int_{\Gamma_{pe}} \underset{1\times 3}{\lfloor p \rfloor} \underset{3\times n_e}{[N]} d\Gamma_{pe}} =$$

$$= \int_{\Gamma_{pe}} \lfloor p_x, p_y, p_z \rfloor \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots & 0 & N_n & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & & 0 & 0 & N_n \end{bmatrix} d\Gamma_{pe}$$

Potential energy of loading in a finite element

local notation:



n – no. of nodes per FE

n_p – no. of nodal parameters per node

no. of degrees of freedom in FE :

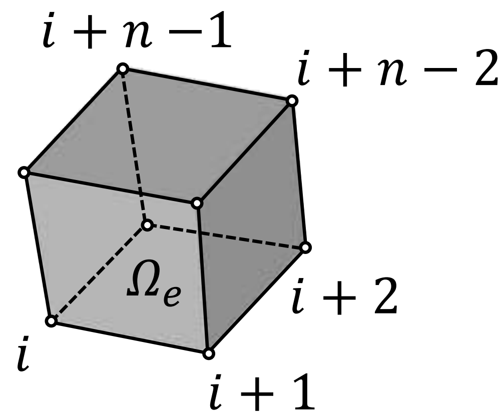
$$n_e = n \cdot n_p$$

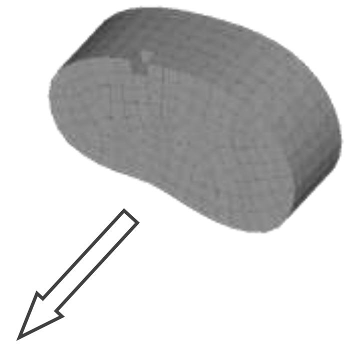
$\underset{n_e \times 1}{\{q\}_e}$ - local vector of nodal parameters

$$W_e = \underset{1\times n_e}{\lfloor q \rfloor_e} \underset{n_e \times 1}{\{F\}_e}$$

equivalent load vector

global notation:





NON – no. of nodes

n_p – no. of nodal parameters per node

no. of degrees of freedom :

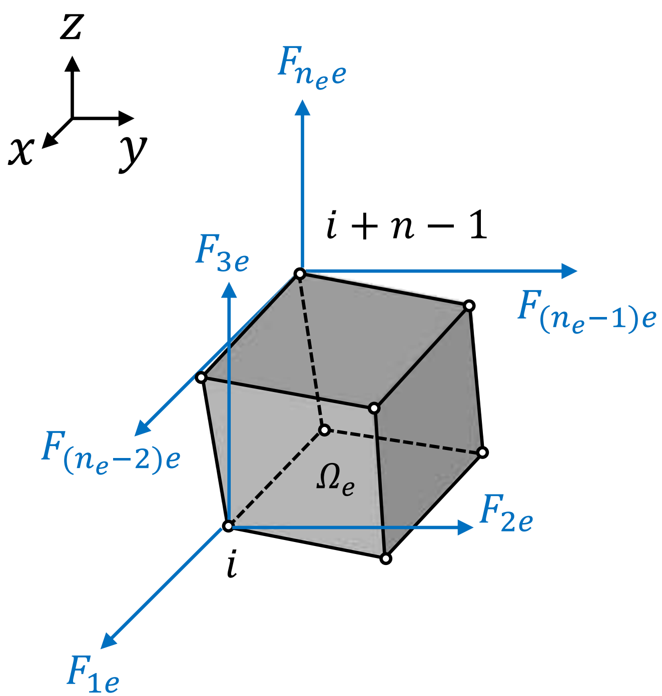
$$NDOF = NON \cdot n_p$$

$\underset{NDOF \times 1}{\{q\}}$ - global vector of nodal parameters

$$W_e = \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times 1}{\{F\}_e^*}$$

extended equivalent load vector

Extended equivalent load vector in a finite element



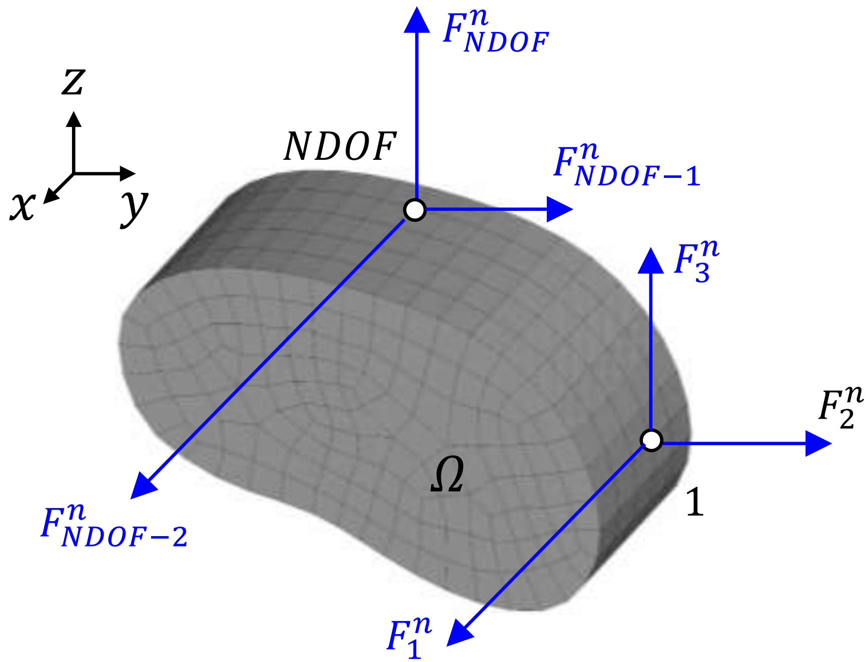
extended equivalent load vector:

$$\underset{NDOF\times 1}{\{F\}_e^*} = \begin{array}{cc} \left\{\begin{matrix} 0 \\ 0 \\ \dots \\ 0 \\ F_{1e} \\ F_{2e} \\ \dots \\ F_{n_e e} \\ 0 \\ \dots \\ 0 \end{matrix}\right\} & \begin{matrix} 1 \\ 2 \\ \\ j-1 \\ j \\ j+1 \\ \\ j+n_e-1 \\ j+n_e \\ \\ NDOF \end{matrix} \end{array}$$

$(assumed\ ascending\ order\ of\ components)$

equivalent load vector: $$\underset{n_e\times 1}{\{F\}_e} = \left\{\begin{matrix} F_{1e} \\ F_{2e} \\ F_{3e} \\ \dots \\ F_{(n_e-2)e} \\ F_{(n_e-1)e} \\ F_{n_e e} \end{matrix}\right\}$$

Forces applied directly on nodes. Potential energy of nodal loads



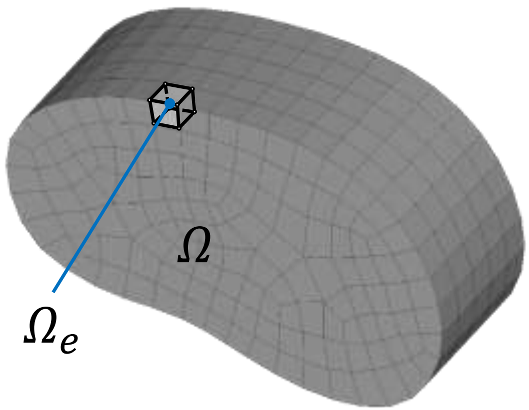
nodal load vector:

$$\underset{NDOF\times 1}{\{F\}^n} = \left\{ \begin{array}{c} F^n_1 \\ F^n_2 \\ F^n_3 \\ \ldots \\ F^n_{NDOF-2} \\ F^n_{NDOF-1} \\ F^n_{NDOF} \end{array} \right\}$$

potential energy of nodal loads:

$$\boxed{W^n = \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF\times 1}{\{F\}^n}}$$

Potential energy of loading in a FE model. Global load vector



potential energy of element loads:

$$\Omega = \sum_{e=1}^{NOE} \Omega_e \quad \rightarrow \quad \boxed{W^e = \sum_{e=1}^{NOE} W_e}$$

NOE – no. of FEs

$NDOF$ – no. of degrees of freedom

potential energy of nodal loads

potential energy of loading in a finite element model: $\boxed{W = W^e + W^n}$

$$W = \sum_{e=1}^{NOE} W_e + W^n = \sum_{e=1}^{NOE} \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF\times 1}{\{F\}_e^*} + \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF\times 1}{\{F\}^n} = \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \Big(\sum_{e=1}^{NOE} \underset{NDOF\times 1}{\{F\}_e^*} + \underset{NDOF\times 1}{\{F\}^n}\Big)$$

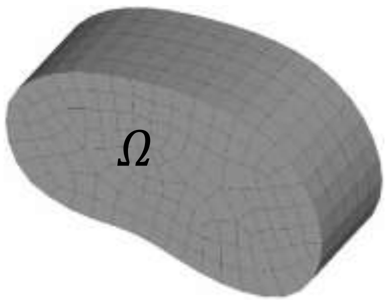
$$= \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \Big(\underset{NDOF\times 1}{\{F\}^e} + \underset{NDOF\times 1}{\{F\}^n}\Big) \quad \rightarrow \quad W = \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF\times 1}{\{F\}}$$

global load vector of element loads

nodal load vector

global load vector: $\boxed{\underset{NDOF\times 1}{\{F\}} = \underset{NDOF\times 1}{\{F\}^e} + \underset{NDOF\times 1}{\{F\}^n}}$

Total potential energy in a FE model. Set of linear equations



NOE – no. of FEs
$NDOF$ – no. of degrees of freedom

Total potential energy of the entire model:

$$V = U - \mathrm{W} = \frac{1}{2} \cdot \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF\times NDOF}{[K]} \cdot \underset{NDOF\times 1}{\{q\}} - \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF\times 1}{\{F\}}$$

$$\underset{NDOF\times 1}{\{q\}} = ?$$

$$V \rightarrow min$$

$$\frac{\partial V}{\partial q_j} = 0 \rightarrow \boxed{\underset{NDOF\times NDOF}{[K]} \cdot \underset{NDOF\times 1}{\{q\}} = \underset{NDOF\times 1}{\{F\}}}$$

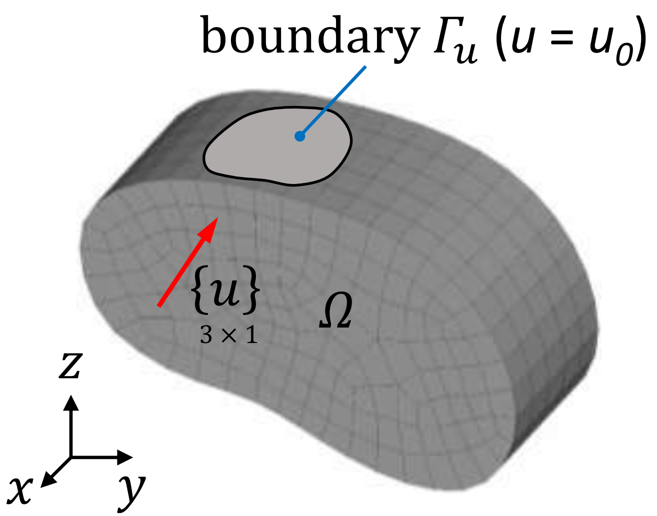
$\uparrow$

set of linear algebraic equations

$$\det\left(\underset{NDOF\times NDOF}{[K]}\right) = 0$$

Set of FE equations with boundary conditions

The displacement field $\{u\}$ that represents solution of the problem fulfils displacement boundary conditons on Γ_u and minimizes the total potential energy V.



$NDOF$ – no. of degrees of freedom

NOF – no. of known degrees of freedom on Γ_u

N – number of unknown degrees of freedom:

$$N = NDOF - NOF$$

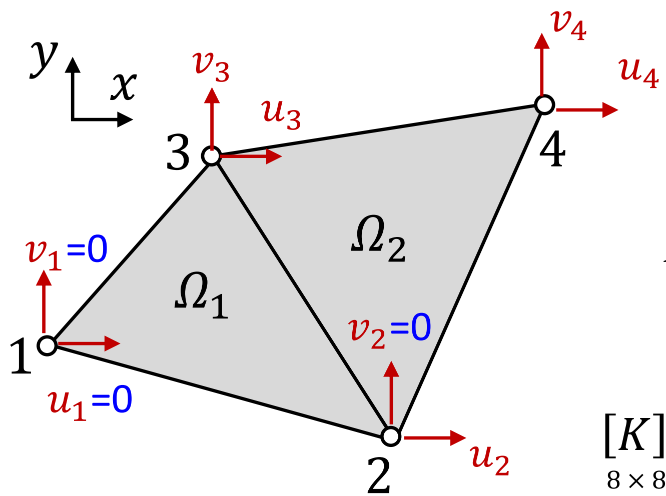
$$\underset{NDOF\times NDOF}{[K]} \rightarrow \underset{N\times N}{[K]} \;;\; \underset{NDOF\times 1}{\{q\}} \rightarrow \underset{N\times 1}{\{q\}} \;;\; \underset{NDOF\times 1}{\{F\}} \rightarrow \underset{N\times 1}{\{F\}}$$

$$\boxed{\underset{N\times N}{[K]}\cdot\underset{N\times 1}{\{q\}} = \underset{N\times 1}{\{F\}}}$$

$$\det\left(\underset{N\times N}{[K]}\right) \neq 0$$

linear set of algebraic equations with boundary conditions

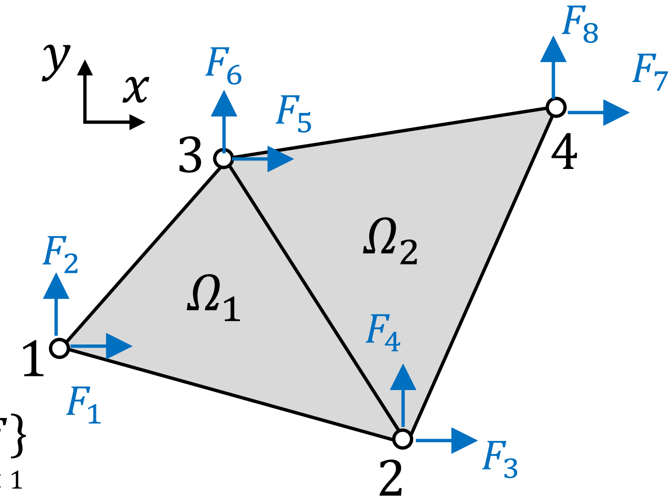
Example 5. Boundary conditions for 2D problem. FE model with two 3-node triangles



$NDOF = 8$

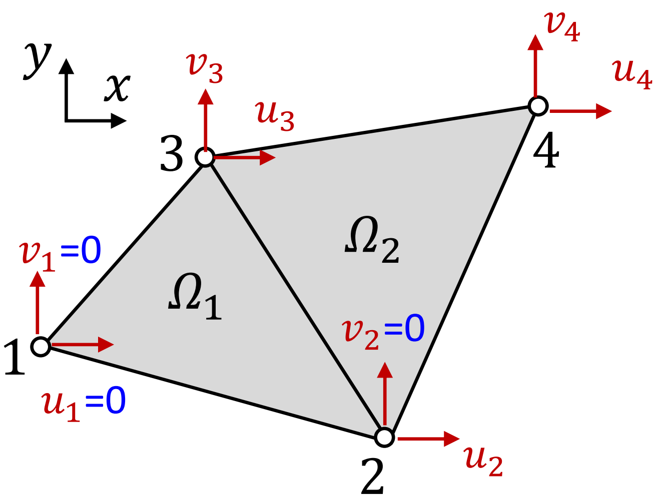
$NOF = 3$

$$\underset{8\times 8}{[K]} \cdot \underset{8\times 1}{\{q\}} = \underset{8\times 1}{\{F\}}$$



$$
\begin{array}{c}
\begin{array}{cc}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\begin{bmatrix}
a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & 0 & 0 \\
b_1 & g_1 & h_1 & i_1 & j_1 & k_1 & 0 & 0 \\
c_1 & h_1 & l_1+a_2 & m_1+b_2 & n_1+c_2 & o_1+d_2 & e_2 & f_2 \\
d_1 & i_1 & m_1+b_2 & p_1+g_2 & r_1+h_2 & s_1+i_2 & j_2 & k_2 \\
e_1 & j_1 & n_1+c_2 & r_1+h_2 & t_1+l_2 & \bar{u}_1+m_2 & n_2 & o_2 \\
f_1 & k_1 & o_1+d_2 & t_1+l_2 & \bar{u}_1+m_2 & \bar{w}_1+p_2 & r_2 & s_2 \\
0 & 0 & e_2 & j_2 & n_2 & r_2 & t_2 & \bar{u}_2 \\
0 & 0 & f_2 & k_2 & o_2 & s_2 & \bar{u}_2 & \bar{w}_2
\end{bmatrix}
\end{array}
\end{array}
\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 = 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}
=
\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}
$$

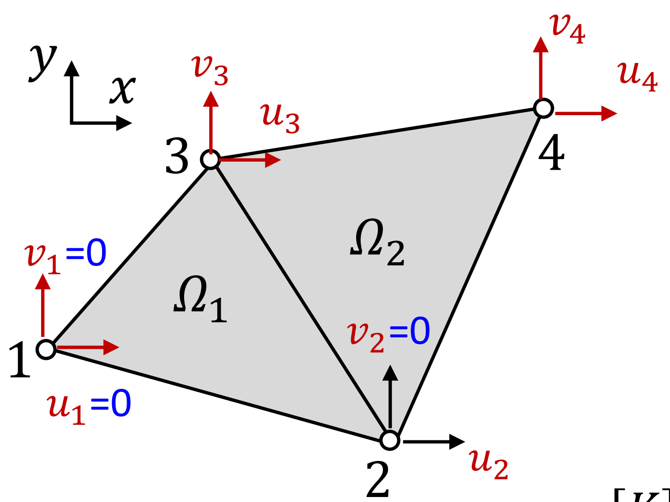
Example 5. Boundary conditions for 2D problem. FE model with two 3-node triangles

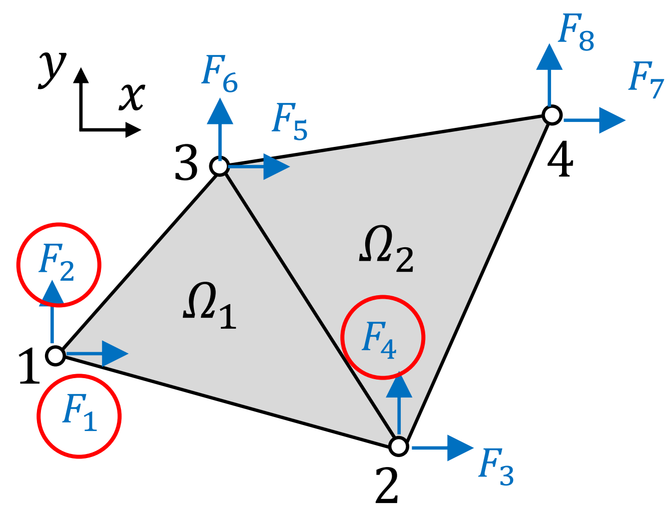


$$\underset{8\times 8}{[K]} \cdot \underset{8\times 1}{\{q\}} = \underset{8\times 1}{\{F\}}$$

$$\begin{array}{c|c}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\begin{bmatrix}
a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & 0 & 0 \\
b_1 & g_1 & h_1 & i_1 & j_1 & k_1 & 0 & 0 \\
c_1 & h_1 & l_1+a_2 & m_1+b_2 & n_1+c_2 & o_1+d_2 & e_2 & f_2 \\
d_1 & i_1 & m_1+b_2 & p_1+g_2 & r_1+h_2 & s_1+i_2 & j_2 & k_2 \\
e_1 & j_1 & n_1+c_2 & r_1+h_2 & t_1+l_2 & \bar{u}_1+m_2 & n_2 & o_2 \\
f_1 & k_1 & o_1+d_2 & t_1+l_2 & \bar{u}_1+m_2 & \bar{w}_1+p_2 & r_2 & s_2 \\
0 & 0 & e_2 & j_2 & n_2 & r_2 & t_2 & \bar{u}_2 \\
0 & 0 & f_2 & k_2 & o_2 & s_2 & \bar{u}_2 & \bar{w}_2
\end{bmatrix}
\end{array}
\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 = 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}
=
\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

Example 5. Boundary conditions for 2D problem. FE model with two 3-node triangles





$N = 8 - 3 = 5$

$$\underset{5\times5}{[K]} \cdot \underset{5\times1}{\{q\}} = \underset{5\times1}{\{F\}}$$

$$\begin{bmatrix} l_1+a_2 & n_1+c_2 & o_1+d_2 & e_2 & f_2 \\ n_1+c_2 & t_1+l_2 & u_1+m_2 & n_2 & o_2 \\ o_1+d_2 & u_1+m_2 & w_1+p_2 & r_2 & s_2 \\ e_2 & n_2 & r_2 & t_2 & \bar{u}_2 \\ f_2 & o_2 & s_2 & \bar{u}_2 & \bar{w}_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

linear set of algebraic equations with boundary conditions

Solution of a set of FE equations with boundary conditions

$$\underset{N\times N}{[K]}\cdot\underset{N\times 1}{\{q\}} = \underset{N\times 1}{\{F\}} \quad\rightarrow\quad \det\left(\underset{N\times N}{[K]}\right)\neq 0 \quad\rightarrow\quad \underset{N\times 1}{\{q\}} = \underset{N\times N}{[K]^{-1}}\,\underset{N\times 1}{\{F\}}$$

$DOF\ solution$: $\underset{NDOF\times 1}{\{q\}}$

$Element\ solution\ (ES)$:

$$\underset{6\times 1}{\{\varepsilon\}} = \underset{6\times n_e}{[B]}\underset{n_e\times 1}{\{q\}_e} \quad ; \quad \underset{6\times 1}{\{\sigma\}} = \underset{6\times 6}{[D]}\,\underset{6\times 1}{\{\varepsilon\}} = \underset{6\times 6}{[D]}\,\underset{6\times n_e}{[B]}\,\underset{n_e\times 1}{\{q\}_e}$$

strain in a finite element

stress in a finite element

$Nodal\ solution\ (NS)$:

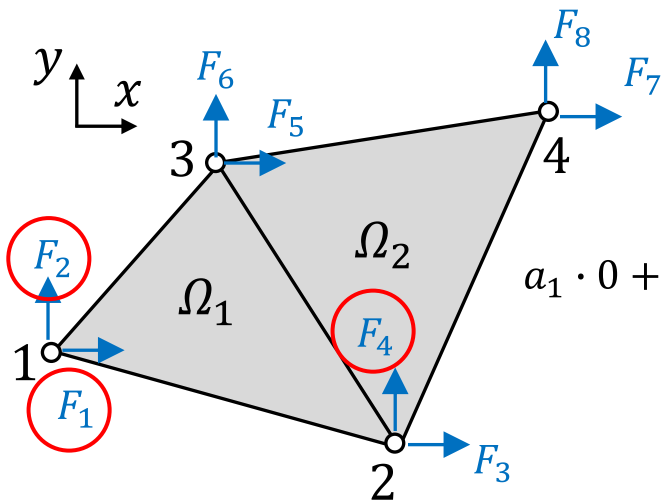
$$\boxed{(NS)_i = \frac{\sum_{e=1}^{k}(ES)_{ei}}{k}}$$

$(NS)_i$ − avaraged nodal solution at node (i)

$(ES)_{ei}$ − element solution in element (e) and at node (i)

k − no. of elements adjacent to node (i)

Example 6. Reactions calculation for 2D problem. FE model with two 3-node triangles



$$\underset{8\times 8}{[K]} \cdot \underset{8\times 1}{\{q\}} = \underset{8\times 1}{\{F\}}$$

known

$$\boxed{\quad\quad} \cdot \boxed{\;} = F_1$$

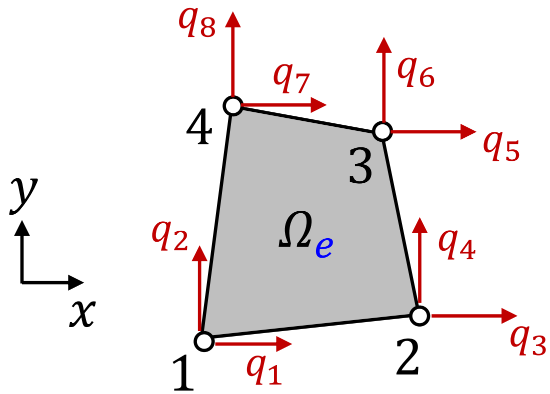
$$a_1 \cdot 0 + b_1 \cdot 0 + c_1 \cdot u_2 + d_1 \cdot 0 + e_1 \cdot u_3 + f_1 \cdot v_3 + 0 \cdot u_4 + 0 \cdot v_4 = F_1$$

$$\boxed{\quad\quad} \cdot \boxed{\;} = F_2 \quad ; \quad \boxed{\quad\quad} \cdot \boxed{\;} = F_4$$

	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	l_1+a_2	m_1+b_2	n_1+c_2	o_1+d_2	e_2	f_2
4	d_1	i_1	m_1+b_2	p_1+g_2	r_1+h_2	s_1+i_2	j_2	k_2
5	e_1	j_1	n_1+c_2	r_1+h_2	t_1+l_2	$\bar{u}_1+m_2$	n_2	o_2
6	f_1	k_1	o_1+d_2	t_1+l_2	$\bar{u}_1+m_2$	$\bar{w}_1+p_2$	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

$$\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 = 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

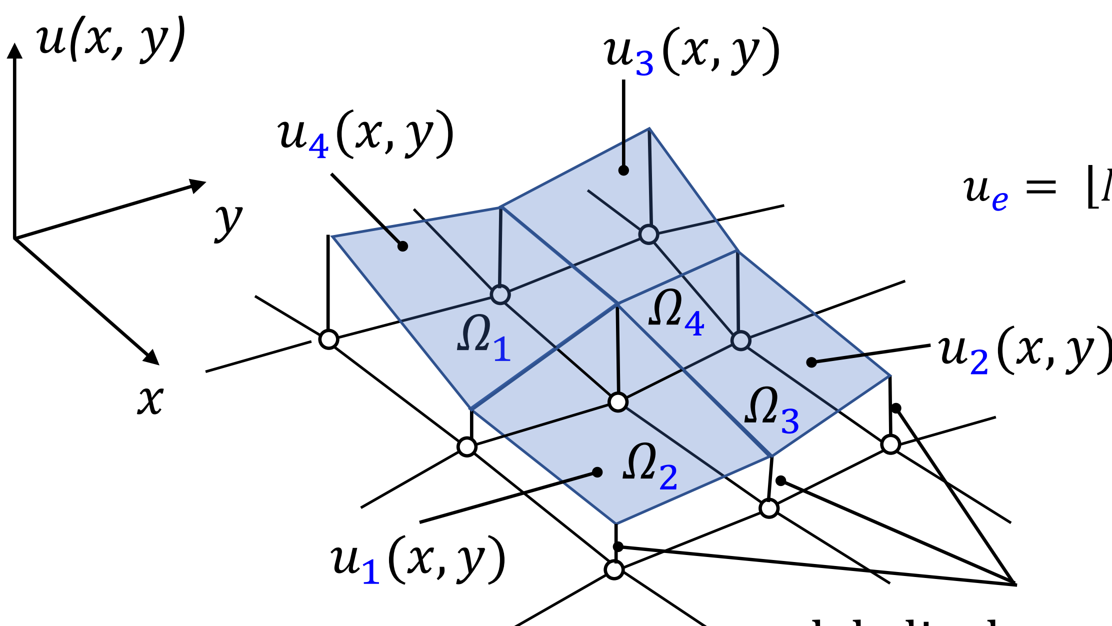
Example 7. DOF solution $u(x,y)$ for 2D problem. FE model with 4-node quadrilateral elements



$$\underset{2\times 1}{\{u\}} = \underset{2\times 8}{[N]}\underset{8\times 1}{\{q\}_e}$$

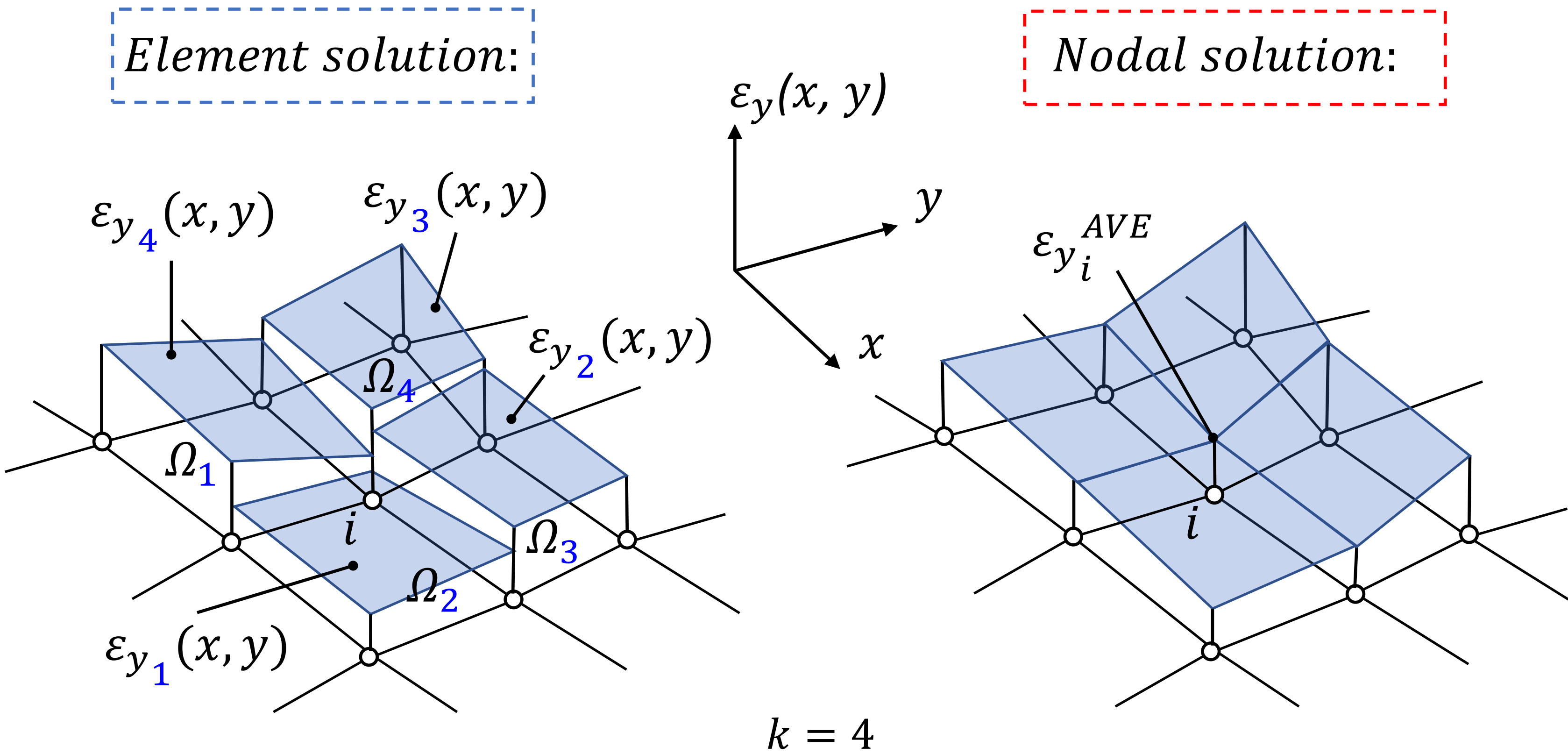
$u_e(x,y)$ – displacement in x direction

$$u_e = \lfloor N_1 \;\; 0 \;\; N_2 \;\; 0 \;\; N_3 \;\; 0 \;\; N_4 \;\; 0 \rfloor \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix}_e$$



nodal displacement in x direction

Example 8. Strain component $\varepsilon_y(x,y)$ for 2D problem. FE model with 4-node quadrilateral elements



$$
{\varepsilon_y}_i^{AVE} = \frac{\varepsilon_{y_1}(x_i, y_i) + \varepsilon_{y_2}(x_i, y_i) + \varepsilon_{y_3}(x_i, y_i) + \varepsilon_{y_4}(x_i, y_i)}{4}
$$

FE modelling – basic steps

Mathematical model

↓

Preprocessor

geometry, element types, material properties, approximation, discretization

↓

Discrete model

↓

Solution

analysis type, boundary conditions, solver, content of a result file

↓ *numerical calculations*

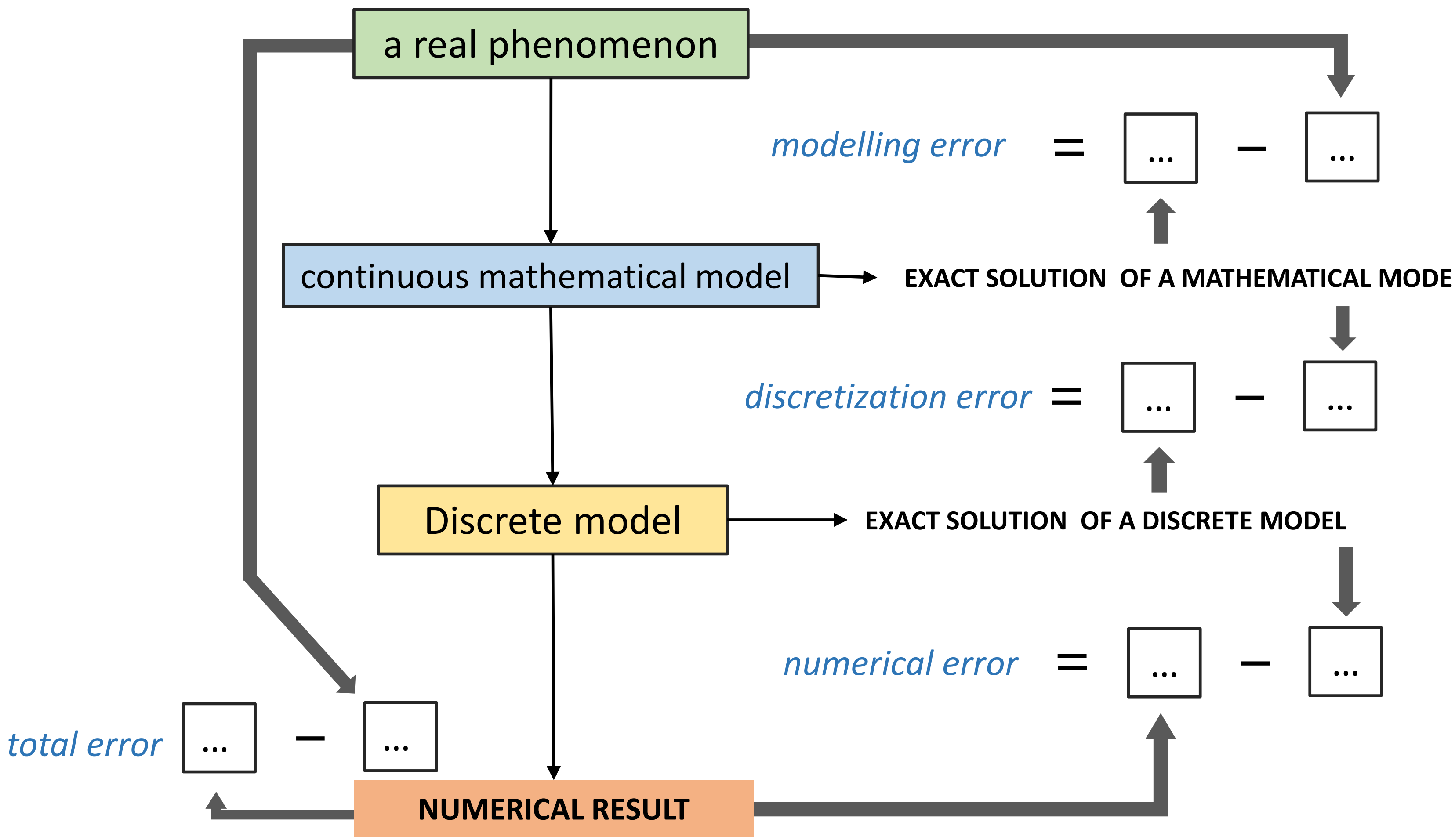
NUMERICAL RESULT

↓

Postprocessor

numbers, graphs, contour maps, animations

Accuracy of FEM calculations



total error = modelling error + discretization error + numerical error

modelling error ≈ discretization error ≈ numerical error → min